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Solutions of Ginzburg-Landau type systems with Higher-dimensional Zero Sets

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1 Introduction

In this paper, we consider the following elliptic system of diagonal type:

$$\Delta V+\lambda\left(1-|V|^{2}\right)V=0 \tag{1}$$

where $V=(V^{1},\cdots,V^{n})$ is defined on some domain in $\mathbf{R}^{n+k}$, $n\geq 2$, $k\geq 1$, and $\lambda\in\mathbf{R}$ is a parameter.

Here, we construct some solutions of (1) on certain domains in $\mathbf{R}^{n+k}$, with boundary values, invariant under the action of a k-parameter group of isometries of $\mathbf{R}^{n+k}$, and having nontrivial k-dimensional zero sets.

When $n=2$, the equation (1) is the Ginzburg-Landau system (GLS), which is used as a mathematical model for many physical phenomena, such as super-conductivity and super-fluidity. In the theory of super-conductivity, the unknown V represents an order parameter which has two degrees of freedom, and its zero set, called *vortices*, corresponds to the region of the normal state in super-conductors. So, especially our result produces an example of solutions of the GLS in $\mathbf{R}^{3}$ with curved vortex lines.

Some results concerning the isolated zeros of solutions of the GLS in $\mathbf{R}^{2}$ are known([1], [2]), however there seems to be no explicit example of solutions with higher-dimensional nontrivial zero sets.

Our proof is based on the "equivariant construction "method due to N.Smale [9], in which the examples of minimal hypersurfaces in Euclidean spaces with higher-dimensional *singularities* are shown. Later, the same method was used to construct examples with higher-dimensional singularities, of harmonic maps [4], and of solutions of a certain non-linear elliptic equation [6].

Main result of this paper can be extended to equations with other type of nonlinearl-ities, but we do not pursue here for simplicity of description.

2 Notations and statement of the main result

We follow the setting of "equivariant construction "method described in the papers [9],[4] and [6]: Let $n\geq 2$, $k\geq 1$ be two integers. Let $\mathcal{U}\subset\mathbf{R}^{k}$ be an open set containing $\{0\}\in\mathbf{R}^{k}$ and assume that there is a C^{∞} group action

$$\Phi: t\in\mathcal{U}\longmapsto\Phi(t)\in\mathrm{Isom}(\mathbf{R}^{n+k}),$$

here Isom($\mathbf{R}^{n+k}$) means the group of isometries of $\mathbf{R}^{n+k}$. We will denote $\Phi(t)$ by G_t.

We define

$$\begin{aligned}\Gamma &= \{G_t(0) : t \in \mathcal{U}\},\\ \tilde{\mathbf{B}}^n &= B_1^n(0) \times \{0\}_k = \{\tilde{x} = (x, 0) \in \mathbf{R}^n \times \mathbf{R}^k, |x| < 1\},\\ \Omega &= \{G_t(\tilde{\mathbf{B}}^n) : t \in \mathcal{U}\}.\end{aligned}$$

So, Γ is the orbit of $\{0\} \in \mathbf{R}^{n+k}$ of the group action Φ, and Ω is the unit n-disc bundle over Γ obtained by moving $\tilde{\mathbf{B}}^n$ along Γ by G_t, $t \in \mathcal{U}$. On the group action Φ, we make the following assumptions: Γ is a properly embedded k-dimensional submanifold in $\mathbf{R}^{n+k}$ and whenever $G_t(0) = 0$, we must have $G_t(\tilde{\mathbf{B}}^n) = \tilde{\mathbf{B}}^n$ for any $t \in \mathcal{U}$, that is, the isotropy group of 0 is the same as the one of $\tilde{\mathbf{B}}^n$. Furthermore, when $G_t = O(t) + v_t$ is the decomposition of the element of Isom($\mathbf{R}^{n+k}$), where $O(t) \in O(n+k)$, the orthogonal group of $\mathbf{R}^{n+k}$, and $v_t \in \mathbf{R}^{n+k}$, we define the group action

$$\Phi_\varepsilon : t \in \mathcal{U} \longmapsto G_t^\varepsilon \in \text{Isom}(\mathbf{R}^{n+k}),$$

and

$$\begin{aligned}\Gamma_\varepsilon &= \{G_t^\varepsilon(0) : t \in \mathcal{U}\} = \left(\tfrac{1}{\varepsilon}\right)\Gamma,\\ \Omega_\varepsilon &= \{G_t^\varepsilon(\tilde{\mathbf{B}}^n) : t \in \mathcal{U}\},\end{aligned}$$

where $G_t^\varepsilon = O(t) + \frac{1}{\varepsilon} v_t$. Note that under the assumption of the group action Φ, Ω_ε is well-defined and then Ω_ε is the unit n-disc bundle over Γ_ε obtained by moving $\tilde{\mathbf{B}}^n$ along Γ_ε by G_t^ε, $t \in \mathcal{U}$. Note also that when $\varepsilon > 0$ is sufficiently small, Ω_ε is close locally the trivial product bundle $B_1^n(0) \times \mathbf{R}^k$ over $\{0\}_n \times \mathbf{R}^k$. Finally, for a map $U : \mathbf{R}^{n+k} \to \mathbf{R}^n$, we denote by $\Gamma(U)$ the set of zeros of U, namely, $\Gamma(U) = \{x : U(x) = 0 \in \mathbf{R}^n\}$.

Now we state the main result of this paper.

__Theorem__ *For any $\lambda \in \mathbf{R}$, there exists an open domain $\tilde{\Omega} \subset \mathbf{R}^{n+k}$ containing Γ, on which there are infinitely many solutions of (1) with boundary values, whose zero set is Γ.*

In the proof of the theorem, we will show that there exists $\bar{\varepsilon} > 0$ sufficiently small, such that for any $0 < \varepsilon < \bar{\varepsilon}$, there is a solution U of

$$\Delta U + \lambda \varepsilon^2 (1 - |U|^2) U = 0 \quad \text{in} \quad \Omega_\varepsilon, \tag{2}$$

$$\Gamma(U) = \Gamma_\varepsilon \tag{3}$$

with a boundary data fixed up to a finite dimensional space, and U is invariant under the action Φ_ε, i.e, $U(G_t^\varepsilon(\tilde{x})) = U(\tilde{x})$ for all $\tilde{x} \in \tilde{\mathbf{B}}^n$ and $t \in \mathcal{U}$.

We will find a solution U of (2) by solving the appropriate fixed point problem. We make essential use of the invariant condition of U, thanks to which, we can think of (2) as a PDE on each fibers of the disc bundle Ω_ε, especially on $\tilde{\mathbf{B}}^n$ for $t = 0$. Note that the nonlinear term of (2) is well controlled when ε is small enough, so we can get a solution as a perturbation of the $\mathbf{R}^n$-valued harmonic function $v_0 : B_1^n(0) \to \mathbf{R}^n, v_0(x) = x$. Taking $\tilde{\Omega} = \varepsilon \cdot \Omega_\varepsilon$, and $V(y) = U\left(\frac{y}{\varepsilon}\right)$ for $y \in \tilde{\Omega}$ will give the desired result. The domain $\tilde{\Omega}$ so obtained, is the bundle over Γ of the n-dimensional discs of radius ε, so looks like locally a thin perturbed tube of radius ε with center axis Γ.

Now we describe our coordination of Ω_ε: For $y \in \Omega_\varepsilon$, there exists $x \in B_1^n(0)$ and $t \in \mathcal{U}$ such that $y = G_t^\varepsilon(\tilde{x})$, then let us denote $F : B_1^n(0) \times \mathcal{U} \to \Omega_\varepsilon$, $F(x,t) = G_t^\varepsilon(\tilde{x})$. we will introduce the local coordinate system by this map, and identify y with (r, θ, t) where (r, θ) are polar coordinates for $x \in B_1^n(0)$. So, functions defined on Ω_ε can naturally be considered as functions on $B_1^n(0)\times\mathcal{U}$ by F. Note for $\varepsilon>0$ sufficiently small, r also is the distance to Γ_ε.

In the sequel we use the following function spaces: For $\nu\in\mathbf{R},\alpha\in(0,1),m=0,1,2,$ define

$$C^{m,\alpha,\nu}(\Omega_\varepsilon;\mathbf{R}^n)=\{u\in C^{m,\alpha}_{loc}(\Omega_\varepsilon\setminus\Gamma_\varepsilon;\mathbf{R}^n):|u|_{m,\alpha,\nu}<+\infty\},$$

where $|\cdot|_{m,\alpha,\nu}$ is the norm

$$|u|_{m,\alpha,\nu}=\sup_{0<s\le 1/2}\left(\sum_{j=0}^{m}|\nabla^j u|_{0,[s,2s]}s^{j-\nu}+\sum_{j=0}^{m}|\nabla^j u|_{(\alpha),[s,2s]}s^{j+\alpha-\nu}\right).$$

Here, ∇ and ∇^2 denote the gradient and Hessian respectively on Ω_ε, and $|\eta|_{0,[s,2s]}$ and $|\eta|_{(\alpha),[s,2s]}$ are the sup norm and the α-th Hölder seminorm of a function(or a section) η on Ω_ε over the set $\{y=y(r,\theta,t)\in\Omega_\varepsilon: s\le r\le 2s\}$. These are Banach spaces under the norm $|\cdot|_{m,\alpha,\nu}$, and if $u\in C^{m,\alpha,\nu}(\Omega_\varepsilon;\mathbf{R}^n)$, then $|u|$ decays like r^ν near Γ_ε.

Furthermore, let us define the closed subspace of $C^{m,\alpha,\nu}(\Omega_\varepsilon;\mathbf{R}^n)$ as

$$C^{m,\alpha,\nu}_G(\Omega_\varepsilon;\mathbf{R}^n)=\{u\in C^{m,\alpha,\nu}(\Omega_\varepsilon;\mathbf{R}^n):u(G^\varepsilon_t(\tilde{x}))=u(\tilde{x})\quad\text{for all } x\in B_1^n(0),\ t\in\mathcal{U}\},$$

that is, maps in $C^{m,\alpha,\nu}$ which are Φ_ε-invariant. We also denote $C^{m,\alpha}_G(\partial\Omega_\varepsilon;\mathbf{R}^n)$ for the space of Φ_ε - invariant boundary data in $C^{m,\alpha}(\partial\Omega_\varepsilon;\mathbf{R}^n)$.

Weighted Hölder spaces like above are now widely used for other nonlinear problems, see [9], [10], [4], [6], [8], [5], [3].

3 Proof of the Theorem

In this section, we seek for a solution of (2) satisfying (3) by the same technique as in [9], [4], [6]: linearization and solving the appropriate fixed point problem. First, we construct the approximate solution. We fix $\varepsilon>0$. Let $v_0:B_1^n(0)\to\mathbf{R}^n$ be the identity map $v_0(x)=x$; so evidently $\Gamma(v_0)=\{0\}\in\mathbf{R}^n$ and $\Delta_{B^n}v_0=0$, where Δ_{B^n} means the Laplace operator on $B_1^n(0)$. Now we define the approximate solution $u_\varepsilon:\Omega_\varepsilon\to\mathbf{R}^n$ by

$$u_\varepsilon(G^\varepsilon_t(\tilde{x}))=v_0(x)\quad\text{for}\quad x\in B_1^n(0),\quad \mathrm{t}\in\mathcal{U}$$

where $\tilde{x}=(x,0)\in\tilde{\mathbf{B}}^n$. By definition of Ω_ε and by our assumption on the group action Φ, u_ε is well-defined and invariant under the action Φ_ε. The zero set of u_ε satisfies $\Gamma(u_\varepsilon)=\Gamma_\varepsilon$.

We wish to find a solution of (2) of the form

$$U(u)=u_\varepsilon+u$$

where the perturbation u is assumed to be invariant under the action Φ_ε and to decay rapidly near Γ_ε, so as to ensure that $\Gamma(U(u))=\Gamma_\varepsilon$.

Let $N(u)$ be the left hand side of (2) for $U(u)$, that is,

$$N(u)=\Delta U(u)+\lambda\varepsilon^2(1-|U(u)|^2)U(u).$$

We make a Taylor expansion of $N(u)$ about $u=0$ to get

$$N(u)=N(0)+Lu+Q(u),$$

where

$$
\begin{aligned}
N(0) &= \Delta u_\varepsilon + \lambda\varepsilon^2(1-|u_\varepsilon|^2)u_\varepsilon, \\
Lu &= \frac{d}{dt}N(tu)|_{t=0} \\
&= \Delta u + \lambda\varepsilon^2\left\{(1-|u_\varepsilon|^2)u - 2(u_\varepsilon\cdot u)u_\varepsilon\right\}, \\
Q(u) &= \int_0^1 (1-t)\frac{d^2}{dt^2}N(tu)dt \\
&= (-2\lambda\varepsilon^2)\int_0^1 (1-t)\left\{|u|^2u_\varepsilon + 2(u_\varepsilon\cdot u)u + 3t|u|^2u\right\}dt \\
&= (-\lambda\varepsilon^2)\left\{|u|^2u_\varepsilon + 2(u_\varepsilon\cdot u)u + |u|^2u\right\},
\end{aligned}
$$

here Δ means the Laplace operator on Ω_ε. Now, if we define the linear operators

$$R = \Delta - \Delta_{B^n}$$

and

$$\xi u = \lambda\varepsilon^2\left\{(1-|u_\varepsilon|^2)u - 2(u_\varepsilon\cdot u)u_\varepsilon\right\},$$

then the equation $N(u) = 0$ can be rewritten as

$$\Delta_{B^n}u = -N(0) - Ru - \xi u - Q(u) \tag{4}$$

which we solve by contraction mapping argument on some weighted Hölder space. Note that if u is invariant under the action Φ_ε, all of the terms in (4) are also Φ_ε-invariant, so we can consider (4) as a PDE on the slice $\tilde{\mathbf{B}}^n$. This is crucial for our subsequent arguments.

To estimate the terms in the right hand side of (4), we need the following lemma due to R.Mazzeo and N.Smale [5].

Lemma1 *Under the local coordination by F, we have*

$$\Delta = \Delta_{B^n} + \Delta_{R^k} + e_1\nabla^2 + e_2\nabla, \tag{5}$$

where Δ and ∇ are the Laplace operator and gradient on Ω_ε, $e_1 \in C^\infty((Sym^2\Omega_\varepsilon)^*)$, $e_2 \in C^\infty(T^*\Omega_\varepsilon)$ are smooth sections and satisfy

$$
\begin{aligned}
&|e_1(x,t)| \le C_0 r\varepsilon, \qquad &|e_2(x,t)| \le C_0\varepsilon, \\
&|e_1|_{(\alpha),[s,2s]}s^\alpha \le C_0 s\varepsilon, \qquad &|e_2|_{(\alpha),[s,2s]}s^\alpha \le C_0\varepsilon
\end{aligned}
$$

for some constant C_0 independent of $\varepsilon > 0$ and $\alpha \in (0,1)$.

For functions u invariant under Φ_ε, the factor Δ_{R^k} in (5) drops out.

Using this lemma, we have

Lemma2 *If $\varepsilon > 0$, $1 < \nu < 2$, and $u \in C_G^{2,\alpha,\nu}(\Omega_\varepsilon;\mathbf{R}^n)$, then $N(0), Ru, \xi u, Q(u)$ are all in $C_G^{0,\alpha,\nu-2}(\Omega_\varepsilon;\mathbf{R}^n)$ and the following estimates hold:*

$$
\begin{aligned}
|N(0)|_{0,\alpha,\nu-2} &\le C_1\varepsilon(1+|\lambda|\varepsilon), \\
|Ru|_{0,\alpha,\nu-2} &\le C_1\varepsilon|u|_{2,\alpha,\nu}, \\
|\xi u|_{0,\alpha,\nu-2} &\le C_1|\lambda|\varepsilon^2|u|_{2,\alpha,\nu}, \\
|Q(u)|_{0,\alpha,\nu-2} &\le C_1|\lambda|\varepsilon^2\left(|u|^2_{2,\alpha,\nu} + |u|^3_{2,\alpha,\nu}\right)
\end{aligned}
$$

for some constant $C_1 > 0$ independent of ε and λ.

Proof Since u_ε and u are Φ_ε-invariant, so are also all terms appeared in the right hand side of (4), and can be considered as functions of $B_1^n(0)$. By definition, the map u_ε satisfies $\Delta_{B^n} u_\varepsilon = 0$, so we have

$$\begin{aligned} N(0) &= \Delta u_\varepsilon + \lambda\varepsilon^2(1-|u_\varepsilon|^2)u_\varepsilon \\ &= (\Delta - \Delta_{B^n})u_\varepsilon + \lambda\varepsilon^2(1-|u_\varepsilon|^2)u_\varepsilon. \end{aligned}$$

Then using Lemma1 and the fact that $|\nabla u_\varepsilon(x)| + |\nabla^2 u_\varepsilon(x)| \le C$ and $|u_\varepsilon(x)| \le 1$ for some constant C independent of ε and $x \in B_1^n(0)$, we have

$$\begin{aligned} |N(0)(x)| &\le |e_1\nabla^2 u_\varepsilon(x)| + |e_2\nabla u_\varepsilon(x)| + |\lambda|\varepsilon^2(1-|u_\varepsilon|^2)|u_\varepsilon| \\ &\le Cs\varepsilon + C\varepsilon + |\lambda|\varepsilon^2 \end{aligned}$$

for $s \le |x| \le 2s$. Taking the supremum over the set $\{x : s \le |x| \le 2s\}$ and multiplying $s^{2-\nu}$, we get

$$\begin{aligned} |N(0)|_{0,[s,2s]}s^{2-\nu} &\le s^{2-\nu}\cdot C\varepsilon(1+|\lambda|\varepsilon) \\ &\le C\varepsilon(1+|\lambda|\varepsilon), \end{aligned}$$

since $1 < \nu < 2$ and $0 < s \le 1/2$. Hölder seminorm estimate for $N(0)$ has the same form, then by taking the supremum over $s \le 1/2$, we have the first estimate of the lemma.

Similarly by Lemma1,

$$Ru = (\Delta - \Delta_{B^n})u = e_1\nabla^2 u + e_2\nabla u,$$

so we have

$$\begin{aligned} |Ru|_{0,[s,2s]}s^{2-\nu} &\le Cs\varepsilon|\nabla^2 u|_{0,[s,2s]}s^{2-\nu} + C\varepsilon|\nabla u|_{0,[s,2s]}s^{1-\nu}\cdot s \\ &\le C\varepsilon\left(|\nabla^2 u|_{0,[s,2s]}s^{2-\nu} + |\nabla u|_{0,[s,2s]}s^{1-\nu}\right), \end{aligned}$$

for $0 < s \le 1/2$. Hölder seminorm estimate is also similar, then taking the supremum over $s \le 1/2$ yields the estimate for Ru.

As for the estimates for ξu and $Q(u)$, by using the basic properties of the Hölder seminorm

$$|\mu + \eta|_{(\alpha)} \le |\mu|_{(\alpha)} + |\eta|_{(\alpha)},$$

and

$$|\mu\eta|_{(\alpha)} \le |\mu|_{(0)}|\eta|_{(\alpha)} + |\mu|_{(\alpha)}|\eta|_{(0)},$$

as in the above computation, we can derive the following bounds:

$$\begin{aligned} |\xi u(x)| &\le C|\lambda|\varepsilon^2|u(x)|, \quad x \in B_1^n(0) && (6) \\ |\xi u|_{(\alpha),[s,2s]} &\le C|\lambda|\varepsilon^2\left(|u|_{0,[s,2s]} + |u|_{(\alpha),[s,2s]}\right), && (7) \\ |Q(u)(x)| &\le C|\lambda|\varepsilon^2\left(|u(x)|^2 + |u(x)|^3\right), \quad x \in B_1^n(0) && (8) \\ |Q(u)|_{(\alpha),[s,2s]} &\le C|\lambda|\varepsilon^2\left(|u|_{0,[s,2s]}|u|_{(\alpha),[s,2s]} + |u|^2_{0,[s,2s]} + |u|^2_{0,[s,2s]}|u|_{(\alpha),[s,2s]}\right). && (9) \end{aligned}$$

If we multiply both sides of (6) and (8) by $s^{2-\nu}$, or of (7) and (9) by $s^{2-\nu+\alpha}$ and take the supremum over $s \le 1/2$, we immediately have

$$\sup_{0<s\le 1/2}\left(|\xi u|_{0,[s,2s]}s^{2-\nu} + |\xi u|_{(\alpha),[s,2s]}s^{2-\nu+\alpha}\right) \le C|\lambda|\varepsilon^2|u|_{0,\alpha,\nu}$$

$$\sup_{0<s\le 1/2}\left(|Q(u)|_{0,[s,2s]}s^{2-\nu}+|Q(u)|_{(\alpha),[s,2s]}s^{2-\nu+\alpha}\right)\ \le\ C|\lambda|\varepsilon^2\left(|u|^2_{0,\alpha,\nu}+|u|^3_{0,\alpha,\nu}\right)$$

which complete the proof of the lemma. □

Now, to find solutions of (4), we first recall the unique solvability result for the linear problem $\Delta_{B^n}u=f$ on $B_1^n(0)$, for $f\in C_G^{0,\alpha,\nu-2}(\Omega_\varepsilon;\mathbf{R}^n)$ with some appropriate boundary conditions.

Let us take the sequence of eigenvalues of $\Delta_{S^{n-1}}$ acting on $C^\infty(S^{n-1};\mathbf{R}^n)$, μ_j, $0=\mu_1\le\mu_2\le\cdots$, (counting multiplicity), $\mu_j\to\infty$, and corresponding sequence of L^2 - normalized eigenmaps $\phi_j\in C^\infty(S^{n-1};\mathbf{R}^n)$ such that $\Delta_{S^{n-1}}\phi_j+\mu_j\phi_j=0$, $j=1,2,\cdots$. Let λ_j and $\lambda_j(-)$ be two real solutions of the equation $\lambda^2+(n-2)\lambda-\mu_j=0$, that is

$$\lambda_j=\frac{2-n}{2}+\sqrt{\frac{(n-2)^2}{4}+\mu_j}\quad\text{and}\quad\lambda_j(-)=\frac{2-n}{2}-\sqrt{\frac{(n-2)^2}{4}+\mu_j}.$$

We now fix ν so that $1<\nu<2$ and choose an positive integer J such that $\lambda_J<\nu<\lambda_{J+1}$.

For this J, we define

$$\Pi_J: L^2(S^{n-1};\mathbf{R}^n)\to\{\phi_1,\phi_2,\cdots,\phi_J\}^\perp$$

be the orthogonal projection.

Then we have :

<u>**Lemma3**</u> *If $f\in C_G^{0,\alpha,\nu-2}(\Omega_\varepsilon;\mathbf{R}^n)$ and $\psi\in C_G^{2,\alpha}(\partial\Omega_\varepsilon;\mathbf{R}^n)$ with $0<\alpha<1$, then there exists a unique $u\in C_G^{2,\alpha,\nu}(\Omega_\varepsilon;\mathbf{R}^n)$ such that*

$$\begin{cases}\Delta_{B^n}u & =f\quad on\ \Omega_\varepsilon\setminus\Gamma_\varepsilon,\\ \Pi_J(u|_{\partial\Omega_\varepsilon}) & =\Pi_J(\psi).\end{cases}\tag{10}$$

Furthermore, we have the estimate

$$|u|_{2,\alpha,\nu}\le C_2\left(|f|_{0,\alpha,\nu-2}+|\psi|_{2,\alpha}\right)$$

for some constant C_2 depending only on α.

<u>Proof</u> The proof of this is done by separation of variables and now quite standard (see [3], [9], [4], [6]), so we make only few comments.

If we write

$$\begin{aligned}u(r,\theta)&=\sum_{j=1}^\infty u_j(r)\phi_j(\theta),\quad u_j(r)=\langle u(r,\cdot),\phi_j(\cdot)\rangle_{L^2(S^{n-1};\mathbf{R}^n)},\\ f(r,\theta)&=\sum_{j=1}^\infty f_j(r)\phi_j(\theta),\quad f_j(r)=\langle f(r,\cdot),\phi_j(\cdot)\rangle_{L^2(S^{n-1};\mathbf{R}^n)},\\ \psi(\theta)&=\sum_{j=1}^\infty \psi_j\phi_j(\theta),\quad \psi_j=\langle\psi,\phi_j\rangle_{L^2(S^{n-1};\mathbf{R}^n)},\end{aligned}$$

then each u_j must be the solution of the following ODE with boundary conditions:

$$\begin{cases}a''(r)+\frac{n-1}{r}a'(r)-\frac{\mu_j}{r^2}=f_j(r),\\ a(1)=\psi_j\quad\text{for}\quad j>J,\\ |a(r)|\le Cr^\nu.\end{cases}$$

By elementary ODE argument, Caffarelli, Hardt and Simon [3] showed that

$$\begin{aligned} u_j(r) &= r^{\lambda_j}\int_0^r s^{1-n-2\lambda_j}\int_0^s \tau^{n-1+\lambda_j} f_j(\tau)d\tau ds, \quad (j=1,2,\cdots,J) \\ u_j(r) &= \psi_j r^{\lambda_j} - r^{\lambda_j}\int_r^1 s^{1-n-2\lambda_j}\int_0^s \tau^{n-1+\lambda_j} f_j(\tau)d\tau ds, \quad (j\geq J+1) \end{aligned}$$

are the unique solutions. Thus the map $\sum_{j=1}^{\infty} u_j\phi_j$ formally solves the equation $\Delta_{B^n}u=f$ on $B_1^n(0)$ with $\Pi_J(u|_{\partial\Omega_\varepsilon})=\Pi_J(\psi)$, and in fact C^2 classical sense on $B_1^n(0)\setminus\{0\}$.

To prove the estimate, note that we are dealing with the system of PDE, but in the same situation this was done in [4] using the local supremum estimates of [8] and the standard Schauder estimates in [7] . □

We now apply Lemma2 and Lemma3 to find fixed points of (4). Fix $\alpha\in(0,1)$ and $\nu\in(1,2)$ as before. For $K>0$ and $\varepsilon>0$, let us define

$$B_{K\varepsilon,\alpha,\nu}=\left\{u\in C_G^{2,\alpha,\nu}(\Omega_\varepsilon;\mathbf{R}^n):|u|_{2,\alpha,\nu}\leq K\varepsilon\right\}.$$

Then we prove

<u>**Lemma4**</u> *For any $\lambda\in\mathbf{R}$, there exists $K>0$ and $0<\bar\varepsilon<1$ such that if $\varepsilon<\bar\varepsilon$, $v\in B_{K\varepsilon,\alpha,\nu}$ and $\psi\in C_G^{2,\alpha}(\partial\Omega_\varepsilon;\mathbf{R}^n)$ satisfying $|\psi|_{2,\alpha}\leq\varepsilon$, then the problem: to find $u\in B_{K\varepsilon,\alpha,\nu}$ such that*

$$\begin{cases} \Delta_{B^n}u &= -N(0)-Rv-\xi v-Q(v) \\ \Pi_J(u|_{\partial\Omega_\varepsilon}) &= \Pi_J(\psi) \end{cases} \tag{11}$$

has a unique solution.

<u>Proof</u> The problem above has a unique solution $u\in C_G^{2,\alpha,\nu}(\Omega_\varepsilon;\mathbf{R}^n)$ by Lemma2 and Lemma3. Furthermore according to Lemma2, Lemma3 and $|v|_{2,\alpha,\nu}\leq K\varepsilon$, we have

$$\begin{aligned} |u|_{2,\alpha,\nu} &\leq C_2\left(|\psi|_{2,\alpha}+|N(0)|_{0,\alpha,\nu-2}+|Rv|_{0,\alpha,\nu-2}+|\xi v|_{0,\alpha,\nu-2}+|Q(v)|_{0,\alpha,\nu-2}\right) \\ &\leq C_2\left(\varepsilon+C_1\varepsilon(1+|\lambda|\varepsilon)+C_1\varepsilon\cdot K\varepsilon+|\lambda|\varepsilon^2\cdot K\varepsilon+|\lambda|\varepsilon^2\left(K^2\varepsilon^2+K^3\varepsilon^3\right)\right) \\ &\leq C_3\left(\varepsilon+|\lambda|\varepsilon^2+K\varepsilon^2+|\lambda|\varepsilon^2\left(K\varepsilon+K^2\varepsilon^2+K^3\varepsilon^3\right)\right) \end{aligned}$$

for some constant $C_3>0$.

So, if we can take K and ε such that

$$C_3\left(\frac{1+|\lambda|\varepsilon}{K}+\varepsilon+|\lambda|\varepsilon^2\left(1+K\varepsilon+K^2\varepsilon^2\right)\right)\leq 1,$$

then the proof will be completed. This can be done as follows: First, fix $K>0$ sufficiently large so that

$$\frac{(1+|\lambda|)}{K}<\frac{1}{2C_3},$$

and then, fix $\bar\varepsilon\in(0,1)$ sufficiently small so that

$$\bar\varepsilon+|\lambda|\bar\varepsilon^2\left(1+K\bar\varepsilon+K^2\bar\varepsilon^2\right)<\frac{1}{2C_3}.$$

□

Now, fix $\psi \in C_G^{2,\alpha}(\partial\Omega_\varepsilon; \mathbf{R}^n)$ so that $|\psi|_{2,\alpha} \le \varepsilon (< \bar{\varepsilon})$. Let us denote $T(v)$ the unique solution of (11) for $v \in B_{K\varepsilon,\alpha,\nu}$. Then, by Lemma4, T defines a self-map of $B_{K\varepsilon,\alpha,\nu}$. To show that T is indeed a contraction, we need

__Lemma5__ *There is a constant $C_4 > 0$ independent of $u, v \in C_G^{2,\alpha,\nu}(\Omega_\varepsilon; \mathbf{R}^n)$, ε and λ such that*

$$|Q(u) - Q(v)|_{0,\alpha,\nu-2} \le C_4 |\lambda| \varepsilon^2 |u - v|_{2,\alpha,\nu} \left[|u|_{2,\alpha,\nu} + |v|_{2,\alpha,\nu} + (|u|_{2,\alpha,\nu} + |v|_{2,\alpha,\nu})^2 \right]$$

holds.

__Proof__ This is obtained quite easily by elementary computation and basic property of Hölder seminorms, if we write

$$Q(u) - Q(v) = \left(-\lambda\varepsilon^2\right)(I_1 + 2I_2 + I_3),$$

where

$$\begin{aligned} I_1 &= [(u - v)\cdot(u + v)]\, u_\varepsilon, \\ I_2 &= (u_\varepsilon \cdot u)(u - v) + [u_\varepsilon \cdot (u - v)]\, v, \\ I_3 &= |u|^2 (u - v) + [(u - v)\cdot(u + v)]\, v. \end{aligned}$$

□

Let $u_1 = T(v_1)$ and $u_2 = T(v_2)$ are the unique solution of (11) for fixed ψ, given by Lemma4. Then by Lemma3, Lemma2 and Lemma5, we have

$$\begin{aligned} &|T(v_1) - T(v_2)|_{2,\alpha,\nu} \\ \le\ & C_2 |R(v_1 - v_2)|_{0,\alpha,\nu-2} + C_2 |\xi(v_1 - v_2)|_{0,\alpha,\nu-2} + C_2 |Q(v_1) - Q(v_2)|_{0,\alpha,\nu-2} \\ \le\ & C_2 C_1 \varepsilon |v_1 - v_2|_{2,\alpha,\nu} + C_2 C_1 |\lambda| \varepsilon^2 |v_1 - v_2|_{2,\alpha,\nu} \\ +\ & C_2 C_4 |\lambda| \varepsilon^2 |v_1 - v_2|_{2,\alpha,\nu} \left[|v_1|_{2,\alpha,\nu} + |v_2|_{2,\alpha,\nu} + (|v_1|_{2,\alpha,\nu} + |v_2|_{2,\alpha,\nu})^2 \right] \\ \le\ & C_5 \left[\varepsilon + |\lambda| \varepsilon^2 + |\lambda| \varepsilon^2 \left(K\varepsilon + K^2 \varepsilon^2 \right) \right] |v_1 - v_2|_{2,\alpha,\nu}. \end{aligned}$$

So if we retake $\bar{\varepsilon}$ small enough so that $C_5 \left[\bar{\varepsilon} + |\lambda| \bar{\varepsilon}^2 + |\lambda| \bar{\varepsilon}^2 \left(K\bar{\varepsilon} + K^2 \bar{\varepsilon}^2 \right) \right] < 1$, the map T defines a contraction on the closed subset of a complete metric space, then it has a fixed point u. Thus we have found a map $U = u + u_\varepsilon$ satisfying (2), at least in $\Omega_\varepsilon \setminus \Gamma_\varepsilon$.

Note that when $1 < \nu < 2$, we can extend $u \in C_G^{2,\alpha,\nu}(\Omega_\varepsilon; \mathbf{R}^n)$ (as thought of a map defined on $B_1^n(0) \setminus \{0\}$) to $0 \in B_1^n(0)$ so that $u(0) = 0$ and $|\nabla u(0)| = 0$, then the map U is indeed a smooth solution of (2) on each fibers of Ω_ε. Moreover if we require $\bar{\varepsilon}$ small enough such that $K\bar{\varepsilon} \le 1/2$, then $|U(x)| \ge |u_\varepsilon(x)| - |u(x)| \ge (1/2)|x|$ for any $x \in B_1^n(0)$, so $\Gamma(U) = \Gamma(u_\varepsilon) = \Gamma_\varepsilon$. As noted earlier, simple rescaling by a factor of ε completes the proof of Theorem. □

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